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The Relativistic Gravitational Field of Spherically Symmetric Extended Bodies

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Introduction

Advancements in satellite technology and cosmological research demand precise relativistic corrections for motion in gravitational fields, particularly for approximately spherical sources. This project analyzes the relativistic gravitational field of spherically symmetric bodies, such as Earth and neutron stars, and examines the motion of objects within these fields.

Objectives

- **Derive the metric and for a spherically symmetric extended body.**
- **Analyze relativistic effects including precession and variations in the speed of light.**
- **Formulate equations of motion for objects within such gravitational fields.**

Methods

- **Use Extended Relativity (ER) to calculate the metric of a gravitational field for a spherically symmetric body.**
- **Decompose the body into spherical shells and integrate contributions from differential mass elements.**
- **Simulate light motion and orbital precession to compare predictions from ER and GR.**

The Metrics for a Single-Point Source

The Schwarzschild metric of GR

$$ds^2 = \left(1 - \frac{2M}{\rho}\right)dt^2 - \left(1 - \frac{2M}{\rho}\right)^{-1} d\rho^2 - \rho^2 d\Omega^2$$

The ER metric

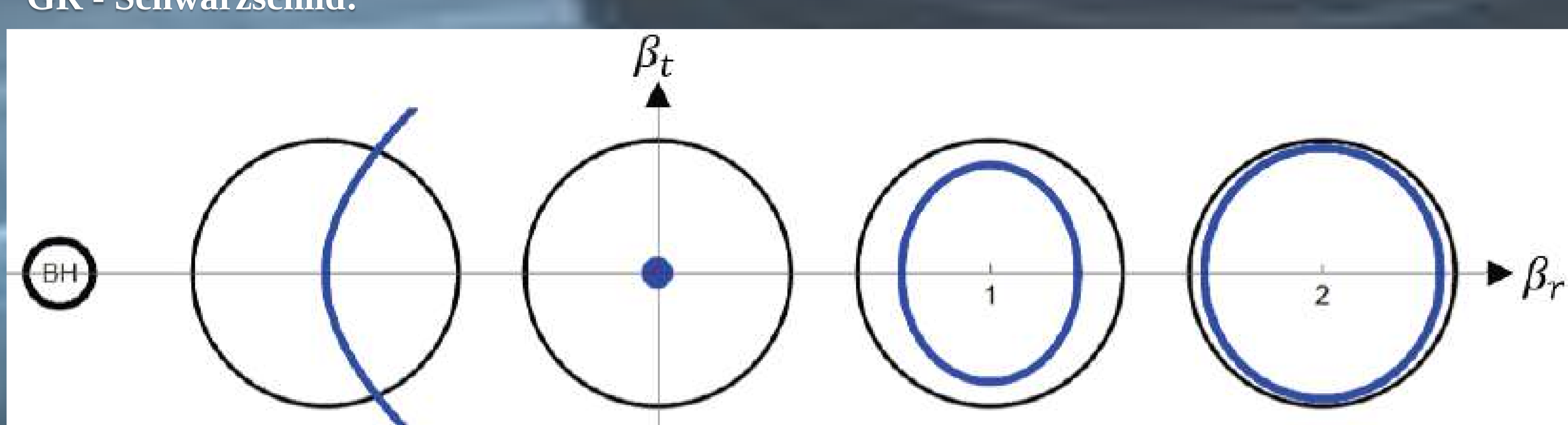
$$ds^2 = \left(1 - \frac{2M}{\rho}\right)dt^2 - \frac{4M}{\rho}dt d\rho - \left(1 + \frac{2M}{\rho}\right)d\rho^2 - \rho^2 d\Omega^2$$

The deviation of the ER metric from flat space-time metric $h_{\mu\nu}(x) = g_{\mu\nu}(x) - \eta_{\mu\nu}$ is linear in M .

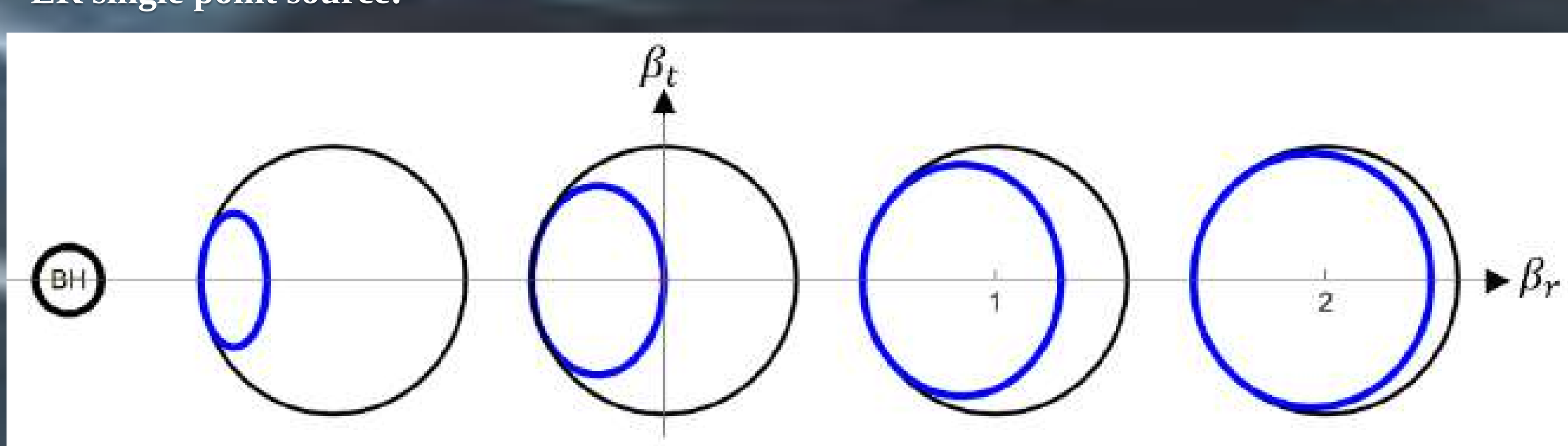
Admissible Velocity Balls

The boundary of the admissible velocity sphere is determined from $ds^2 = 0$ (light). 2D sections of light velocity (blue) near the source are shown for distances r = 1/3 ,1, 3 and 9 Schwarzschild radii. The black circles represent the light velocity in empty space.

GR - Schwarzschild:



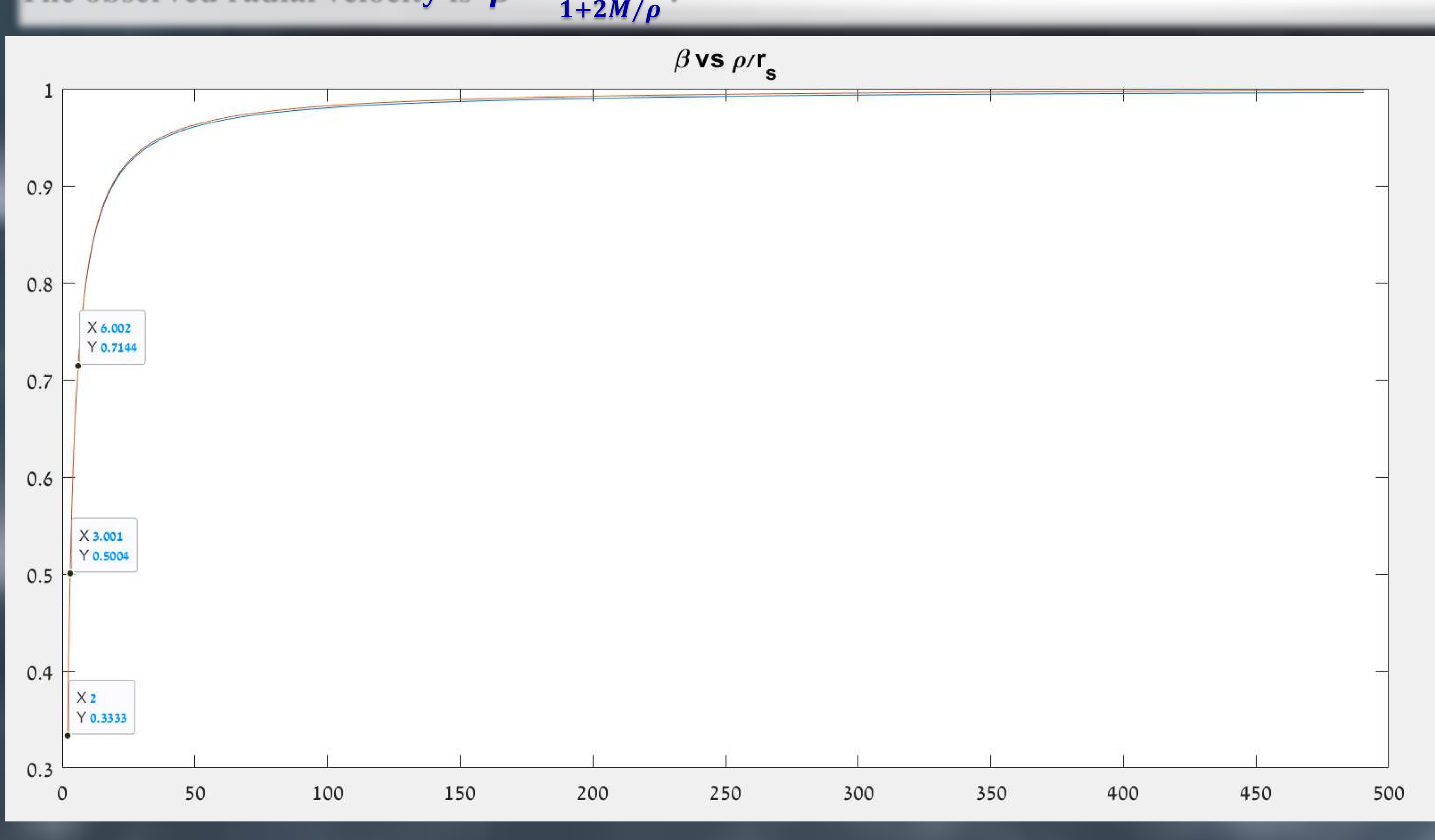
ER single point source:



The Speed of Light Relative to a Far Observer

Light moving away from a black hole, starting beyond the Schwarzschild radius, is observed to accelerate gradually until reaching the speed of light in a vacuum, c .

The observed radial velocity is $\beta = \frac{1-2M/\rho}{1+2M/\rho}$.



Superposition **– The combined field's deviation tensor $h_{\mu\nu}$ is the sum of the deviation tensors from individual sources.**

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = \left(\eta_{\mu\nu} - \sum_j h_{j:\mu\nu}(x)\right)dx^\mu dx^\nu$$

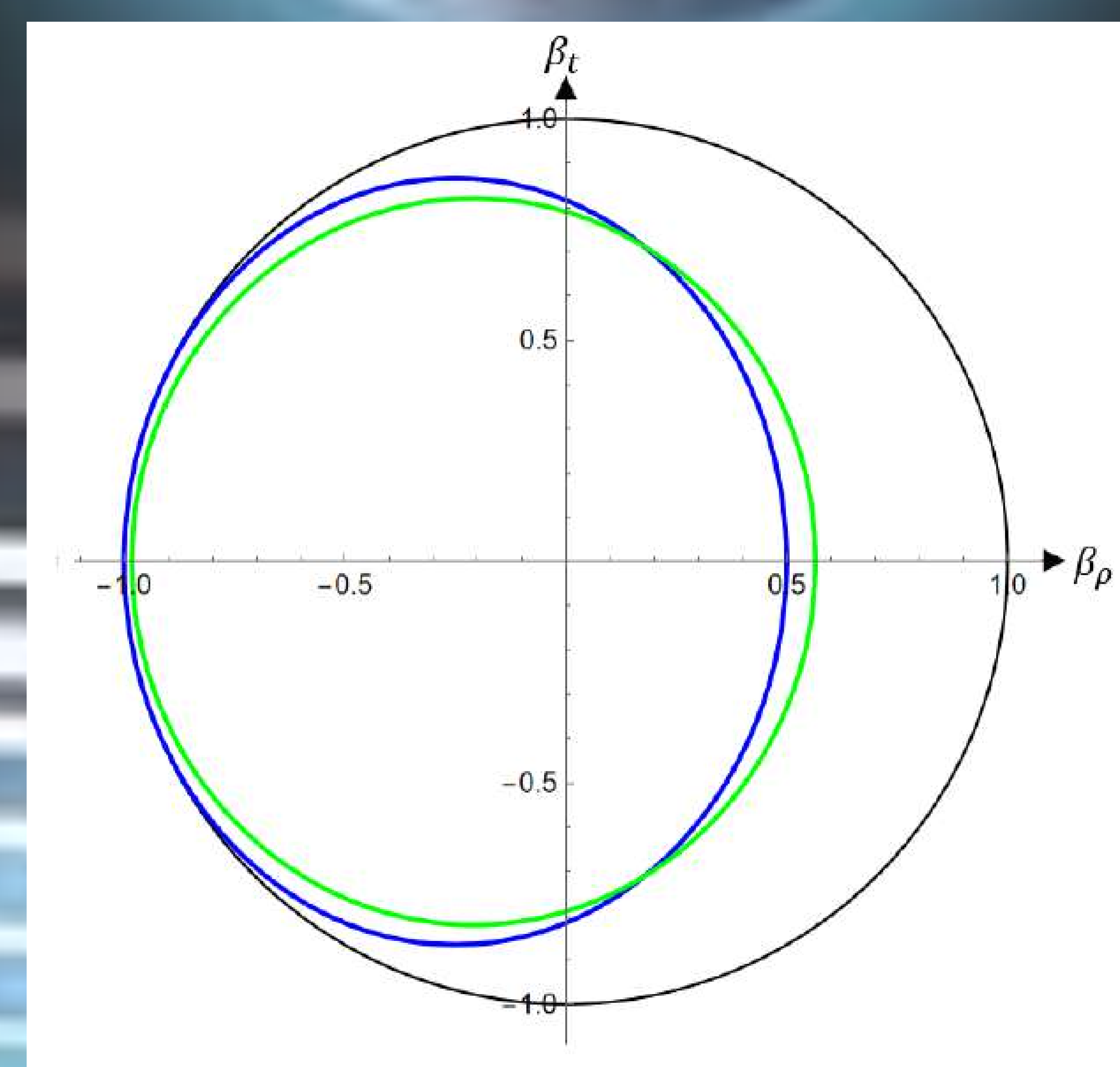
Extended Body Metric **– Derive the explicit formula for the metric of an extended spherically symmetric body's gravitational field.**

$$ds^2 = (1-\phi)dt^2 - 2(\phi - \chi)dt d\rho - (1 + \phi - \psi)d\rho^2 - \left(1 + \frac{\psi}{2}\right)\rho^2 d\Omega^2$$

where:

$\phi = \frac{2M}{\rho}$, $\Psi = \frac{4}{3\rho^3}M_4$, $\chi(\rho) = \sum_{n=0}^{\infty} \frac{2}{(2n+1)(2n+3)\rho^{2n+3}} M_{2n+4}$

Difference in Velocity Ball at the Surface of a Neutron Star **- The velocity ball at distance $\rho = 3r_s$ from the source is shown for the following cases: (1) No gravitational source (black). (2) Single-point source (blue), (3) Extended body with a radius $R = 3r_s$ (green).**



Dilation due to Gravitational Field **- The round-trip time from Earth to the ISS is 2.67 ms without gravitational dilation. Using the extended body model, the gravitational delay is 2.9 ps, while the single-point model predicts a delay of 3.6 ps.**

Relativistic Orbital Precession **- The relativistic orbital precession of Mercury, considering the Sun as a single point source, is $\Delta\varphi = 5.013 \cdot 10^{-7}\, rad$. The additional precession, considering the Sun as an extended body, is $5.262 \cdot 10^{-12}\, rad$.**

Gravitational Field Tensor **– The tensor is as for single point with higher mass moments corrections.**

$$G^0 = -\frac{M}{\rho^2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & -B(\rho) & 0 \\ 0 & 0 & A(\rho) \end{pmatrix}, G^1 = -\frac{M}{\rho^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \frac{M_4}{\rho^4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, G^2 = -\frac{M_4}{\rho^4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

where:

$A(\rho) = \sum_{n=0}^{\infty} \frac{2}{(2n+1)(2n+3)\rho^{2n+3}} M_{2n+4}$, $B(\rho) = \sum_{n=0}^{\infty} \frac{2}{(2n+1)\rho^{2n+3}} M_{2n+4}$

Equation of motion **- The acceleration obtained from the geodesic equation using the metric and the gravitational field tensor.**

$$\ddot{x}^\alpha = \ddot{x}^\alpha_{(q)} - \ddot{x}^0_{(q)}\dot{x}^\alpha$$

where $\dot{x} = \frac{dx}{dt}$, $\ddot{x}^\alpha_{(q)} = \left((I - H)^{-1}\right)^\alpha_\lambda G^\lambda_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$.

The acceleration includes a single-point source term, linear and quadratic mass-dependent terms, and minor higher-order corrections.

Gravitational Field Inside a Spherical Sell **- In ER, inside the spherical shell, there are nonzero terms in the deviation matrix $H^\beta_\mu(x)$, mainly due to gravitational time dilation.**

Discussion and Conclusions

We calculated the metric for a spherically symmetric extended body to examine relativistic phenomena. For Earth and the solar system, the differences from the single-source model are too small to be measurable with current technology and have not been observed in existing measurements. However, it may be possible to design experiments, such as those based on the Doppler shift, to detect these differences. For massive objects like neutron stars, the differences are larger and potentially observable.

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